Math 131 B, Lecture 1
Real Analysis

Sample Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ____________________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1. *10pts.*

Let (S, d_S) and (T, d_T) be two metric spaces, each having more than one point. Their Cartesian product is the set $S \times T = \{(s,t) : s \in S, t \in T\}$. Below are two proposed metrics for $S \times T$. Which is a valid metric? Justify your answer.

$$d_1((s_1,t_1),(s_2,t_2)) = d_S(s_1,s_2) + d_T(t_1,t_2)$$
$$d_2((s_1,t_1),(s_2,t_2)) = d_S(s_1,s_2) \cdot d_T(t_1,t_2)$$

Problem 2.

(a) [5pts.] Let (M, d) be a metric space, and $S \subseteq M$. Give a definition of a limit point of S.

(b) [5pts.] We say that $S \subset M$ is a dense subset of M if every open set in M contains a point of S. Prove that if S is dense in M, $\overline{S} = M$.

Problem 3.

(a) [5pts.] Give a definition of a compact set.

(b) [5pts.] Let S be compact and $X \subset S$ be closed. Prove that X is also compact.

Problem 4.

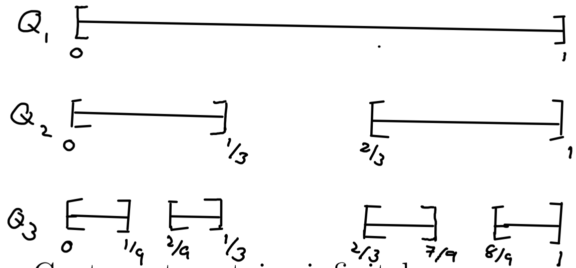
(a) [5pts.] Let $\{\mathbf{x}^k\}$ be a sequence in $\mathbb{R}^n$, where each $\mathbf{x}^k = (x_1^k, \cdots, x_n^k)$. Prove that $\{\mathbf{x}^k\}$ converges in $\mathbb{R}^n$ if and only if each sequence $\{x_i^k\}$ converges in $\mathbb{R}$.

(b) [5pts.] Use part (a) to give a short proof that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. (Hint: Pick a subsequence that converges in the first coordinate, then look at the second coordinate...)

Problem 5.

(a) [5pts.] State the Cantor Intersection Theorem.

(b) [5pts.] The *Cantor set* is a subset of the real line constructed as follows: Let $Q_1 = [0,1]$ and Q_2 be obtained from Q_1 by removing the middle third of the interval, i.e. $Q_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then to obtain Q_3, we cut out the middle thirds of the two remaining intervals, so that $Q_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. We continue to construct Q_n by removing the middle third of each interval in Q_{n-1}. The Cantor set is $\bigcap_{i=1}^{\infty} Q_i$. The first three stages are drawn below.



Prove that the Cantor set contains infinitely many points. [Hint: The quickest way to do this is to prove that the Cantor set has more than 2^n points for any n.]